A Survey of Deep Learning for Mathematical Reasoning

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https://github.com/lupantech/dl4math

Introduction: DL4MATH

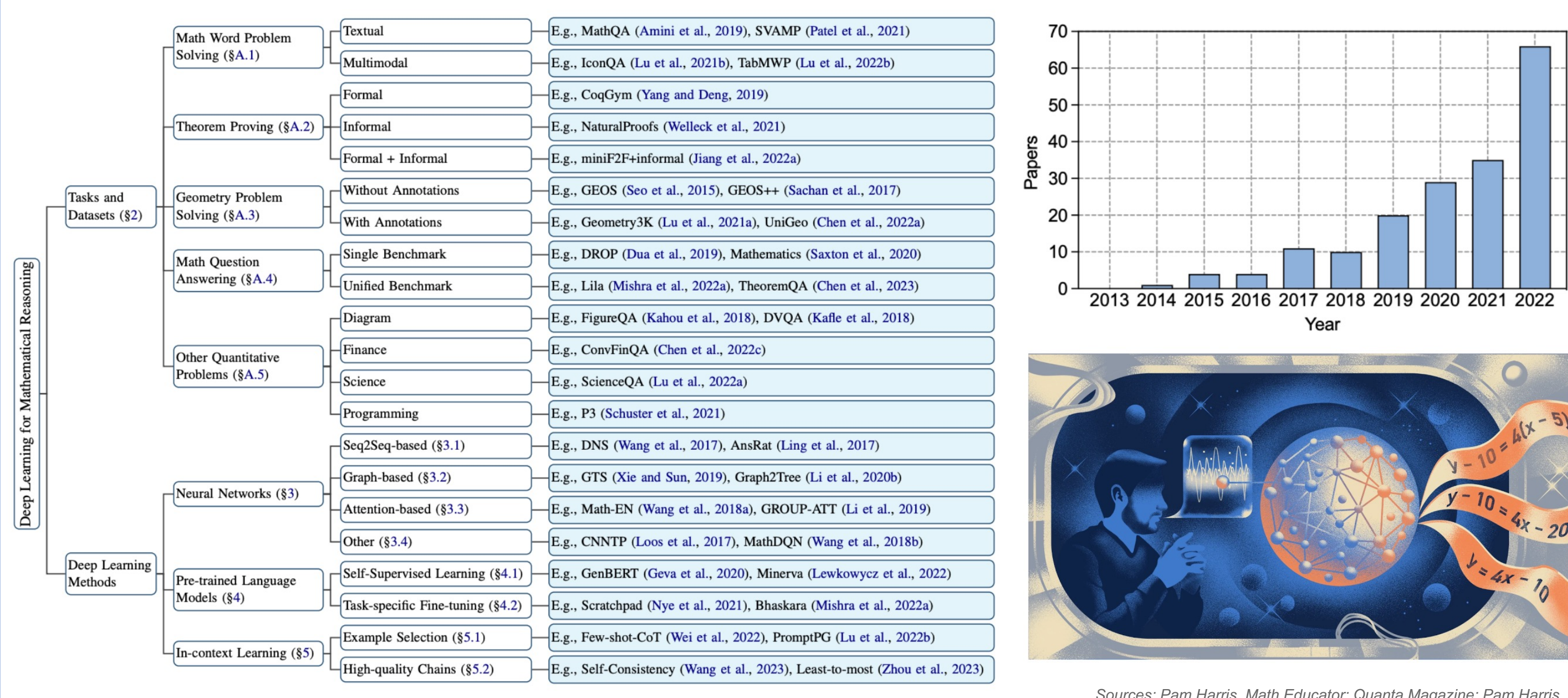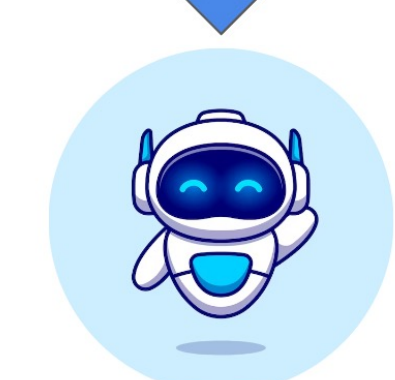Sources: Pam Harris, Math Educator; Quanta Magazine; Pam Harris

Math Word Problems

Automatically solve math word problems

Question: Tom has **2** apples and Jerry has **5** apples. How many apples do they have in **total**?

Rationale: x = **2 + 5**

Solution: 7

Four basic arithmetic operations with single or multiple operation steps.

Question: Sara picked 45 pears and Sally picked 11 pears from the pear tree. How many pears were picked in total?

Math ability: basic math
Language complexity: simple language
Format: generative question answering
Knowledge: no external information
Instruction: You are given a question that involves the calculation of numbers. You need to perform either an addition or subtraction operation on the numbers. Generate your answer to the given question.

Program 1:
```
def solution(x, y):
    answer = x + y
    return answer
print(solution(45, 11)) # total pears is the sum of
pears with Sara and Sally
```
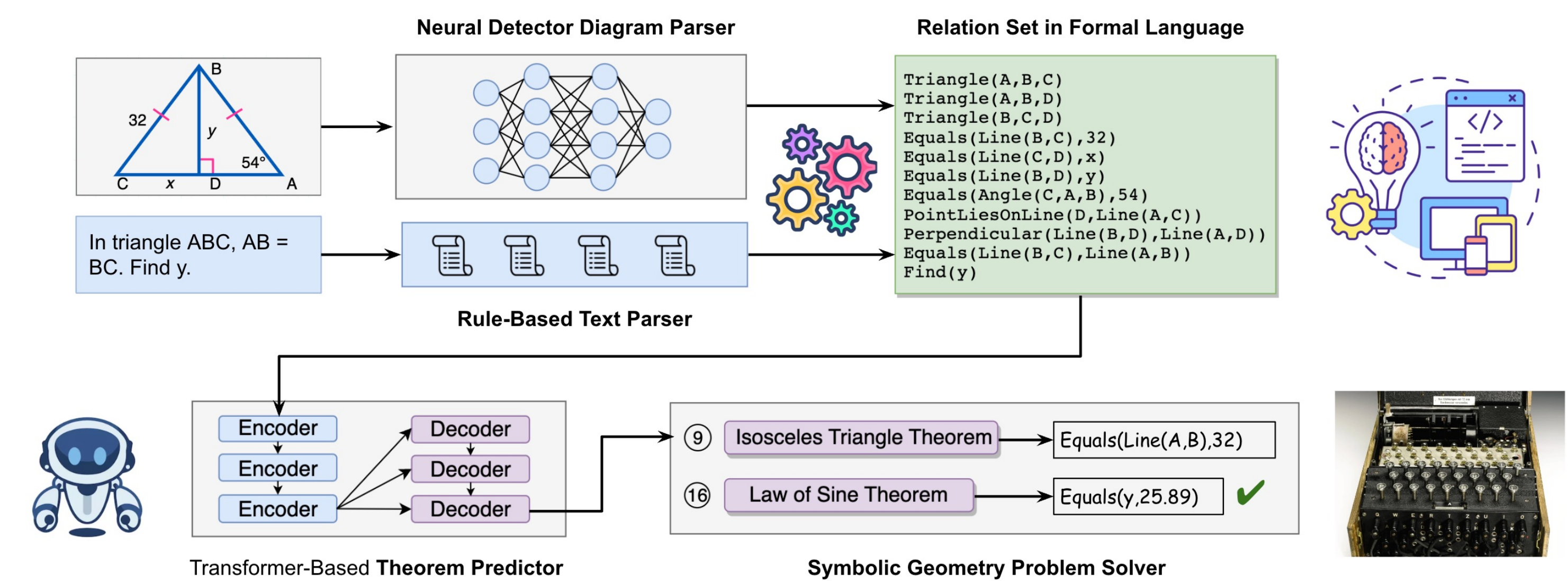
Program 2:
```
x = 45
y = 11
answer = x + y # total pears is the sum of pears with
Sara and Sally
print(answer)
```

Answer: 56

Lila: A Unified Benchmark for Mathematical Reasoning, EMNLP 2022

Geometry Problem Solving

Neuro-symbolic reasoning over geometry diagrams, theorems, and solvers

Inter-GPS: Interpretable Geometry Problem Solving with Formal Language and Symbolic Reasoning, ACL 2021

Automated Theorem Proving

Demonstrate the truth of a mathematical claim (a **theorem**) via a sequence of logical arguments (a **proof**)

Input: theorem

Statement
If $\gcd(n, 4) = 1$ and $\mathrm{lcm}(n, 4) = 28$, show that n is 7.

Informal proof
We know that $\gcd(a, b) \cdot \mathrm{lcm}(a, b) = ab$, hence $1 \cdot 28 = n \cdot 4$.
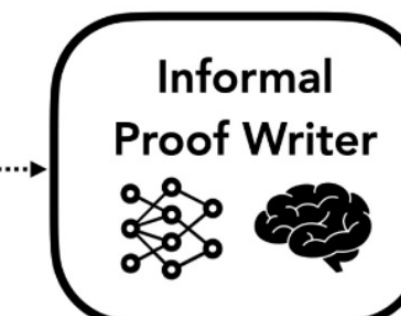Then $n = 1 \cdot 28/4 = 7$, completing the proof. ∎

Output: proof

Formal sketch
```
have c1: "1*28 = n*4"
  using assms
  <proof>
then have c2: "n = 1*28/4"
  <proof>
then show ?thesis
  <proof>
```
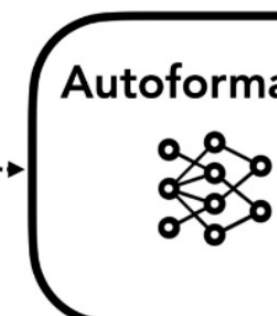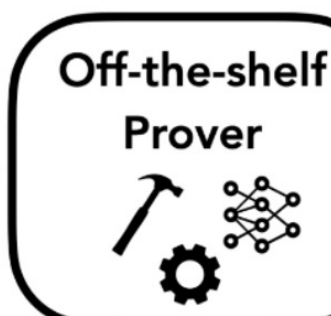
Verified formal proof
```
have c1: "1*28 = n*4"
  using assms
  by (smt (z3) prod_gcd_lcm_nat)
then have c2: "n = 1*28/4"
  by auto
then show ?thesis
  by auto
```

Draft informal proof → **Generate** formal **sketch** → **Prove** remaining gaps

Draft, Sketch, and Prove: Guiding Formal Theorem Provers with Informal Proofs, ICLR 2023

Probing Human-level Intelligence

Probing **Numerical Commonsense** Knowledge

Birds can [MASK]. → **BERT-Large** Masked Word Prediction → 1st: fly (79.5%) 2nd: sing (9.1%)

However, for **Numerical Commonsense** Knowledge:

A bird usually has [MASK] legs. → 1st: four (44.8%) 2nd: two (18.7%)

A car usually has [MASK] wheels. → 1st: four (53.7%) 2nd: two (20.5%)

A car usually has [MASK] round wheels. → 1st: two (37.1%) 2nd: four (20.2%)

Measuring **High-level** Problem Solving

Problem: Tom has a red marble, a green marble, a blue marble, and three identical yellow marbles. How many different groups of two marbles can Tom choose?
Solution: There are two cases here: either Tom chooses two yellow marbles (1 result), or he chooses two marbles of different colors ($\binom{4}{2} = 6$ results). The total number of distinct pairs of marbles Tom can choose is $1 + 6 = \boxed{7}$.

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Problem: The equation $x^2 + 2x = i$ has two complex solutions. Determine the product of their real parts.
Solution: Complete the square by adding 1 to each side. Then $(x+1)^2 = 1 + i = e^{\frac{i\pi}{4}}\sqrt{2}$, so $x + 1 = \pm e^{\frac{i\pi}{8}}\sqrt[4]{2}$. The desired product is then $\left(-1 + \cos\left(\frac{\pi}{8}\right)\sqrt[4]{2}\right)\left(-1 - \cos\left(\frac{\pi}{8}\right)\sqrt[4]{2}\right) = 1 - \cos^2\left(\frac{\pi}{8}\right)\sqrt{2} = 1 - \frac{\left(1 + \cos\left(\frac{\pi}{4}\right)\right)}{2}\sqrt{2} = \frac{1 - \sqrt{2}}{2}$
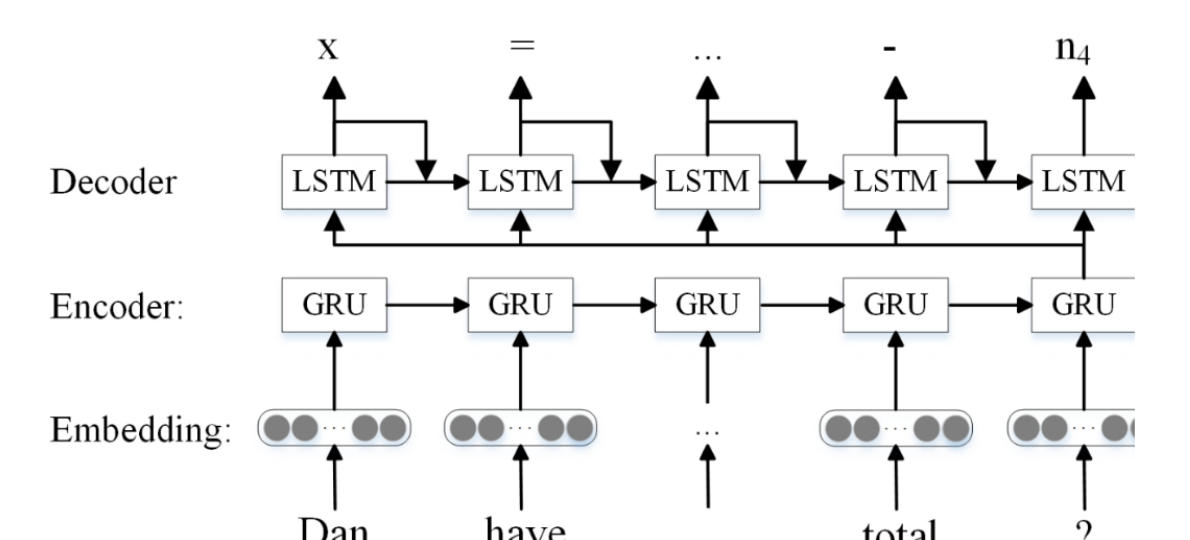
$$i = \sqrt{-1}$$

Birds have four legs?! NumerSense: Probing Numerical Commonsense Knowledge of Pre-trained Language Models, EMNLP 2020
Measuring Mathematical Problem Solving With the MATH Dataset, NeurIPS 2021

Seq2Seq Neural Networks
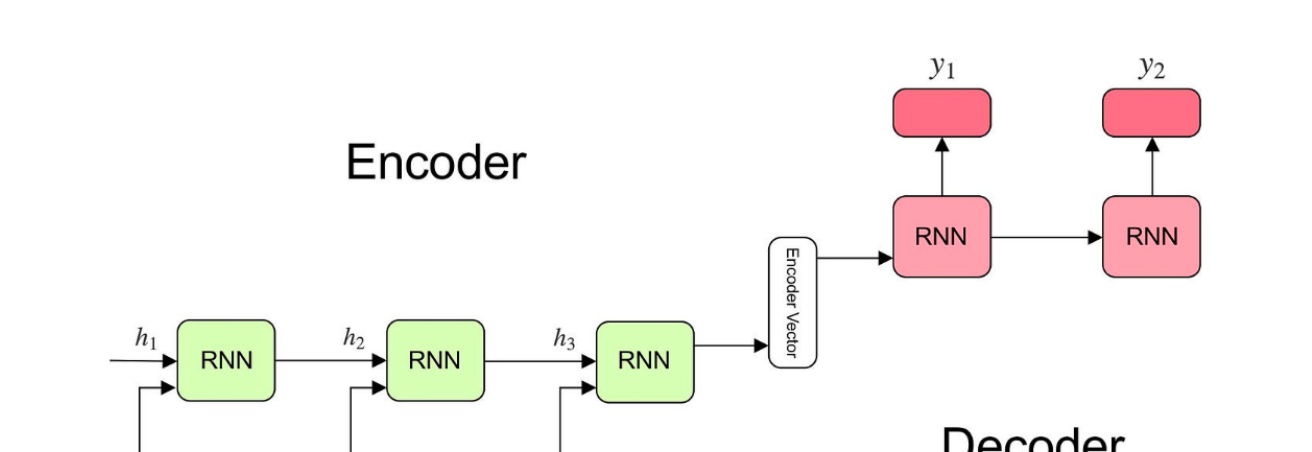
Equation: x = 5 + 4 + 3 - 2; solution: [10]

Model output (Equation template): x = n1 + n3 + n2 - n4

Applying number mapping to equation form

Model input: Dan have n1, pens and n2 pencils, Jessica have n3 more pens and n4 less pencils than him. How many pens and pencils do Jessica have in total?

Problem: Dan have 5 pens and 3 pencils, Jessica have 4 more pens and 2 less pencils than him. How many pens and pencils do Jessica have in total?
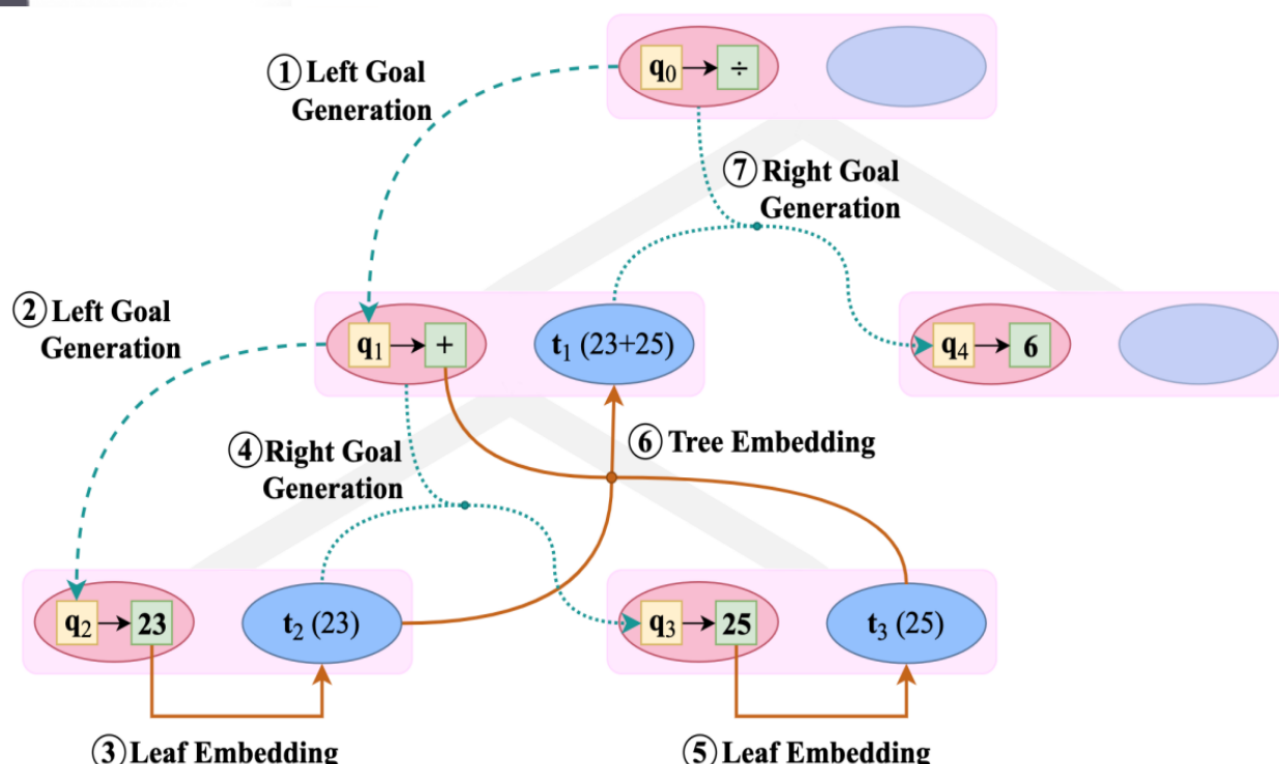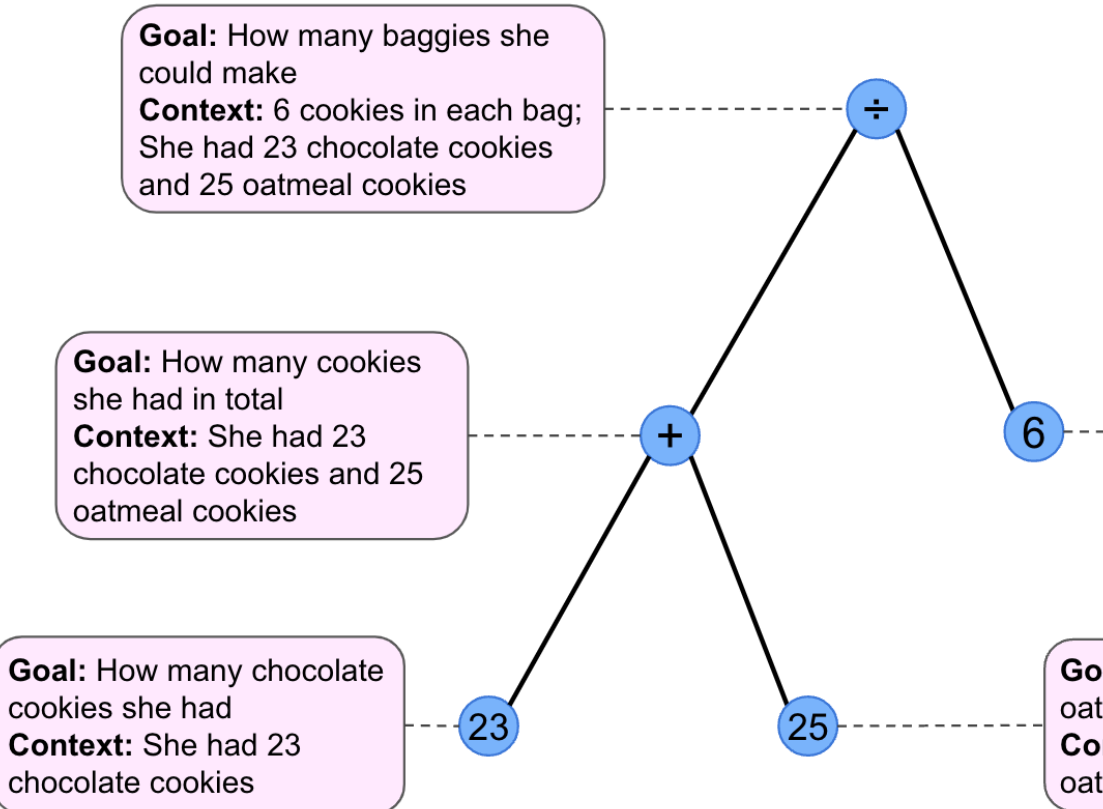
Number mapping: {n1=5, n2=3, n3=4, n4=2}

Deep Neural Solver for Math Word Problems, EMNLP 2017

Tree-based Neural Networks

Problem: Robin was making baggies of cookies with 6 cookies in each bag. If she had 23 chocolate cookies and 25 oatmeal cookies, how many baggies could she make?

Solution Expression: (23 + 25) ÷ 6
Solution: 8

A Goal-Driven Tree-Structured Neural Model for Math Word Problems, IJCAI 2019

LLMs with Chain-of-Thought Prompting

Question: The cafeteria had 23 apples. If they used 20 to make lunch and bought 6 more, how many apples do they have?

Standard Prompting

Model Input
Q: Roger has 5 tennis balls. He buys 2 more cans of tennis balls. Each can has 3 tennis balls. How many tennis balls does he have now?
A: The answer is 11.
Q: The cafeteria had 23 apples. If they used 20 to make lunch and bought 6 more, how many apples do they have?

Model Output
A: The answer is 27. ✗

Chain-of-Thought Prompting

Model Input
Q: Roger has 5 tennis balls. He buys 2 more cans of tennis balls. Each can has 3 tennis balls. How many tennis balls does he have now?
A: Roger started with 5 balls. 2 cans of 3 tennis balls each is 6 tennis balls. 5 + 6 = 11. The answer is 11.
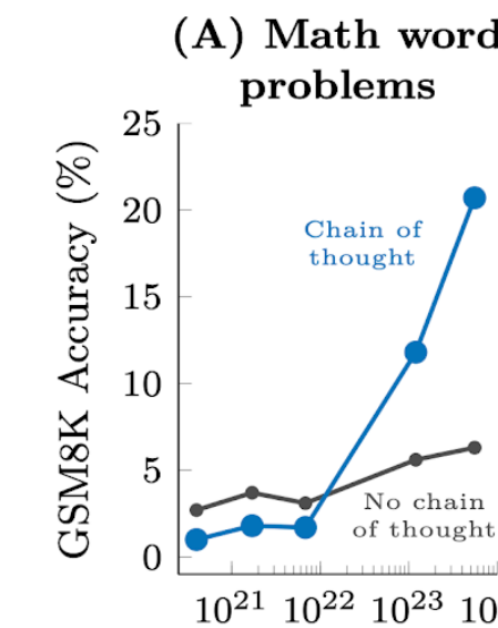Q: The cafeteria had 23 apples. If they used 20 to make lunch and bought 6 more, how many apples do they have?

Model Output
A: The cafeteria had 23 apples originally. They used 20 to make lunch. So they had 23 - 20 = 3. They bought 6 more apples, so they have 3 + 6 = 9. The answer is 9. ✓

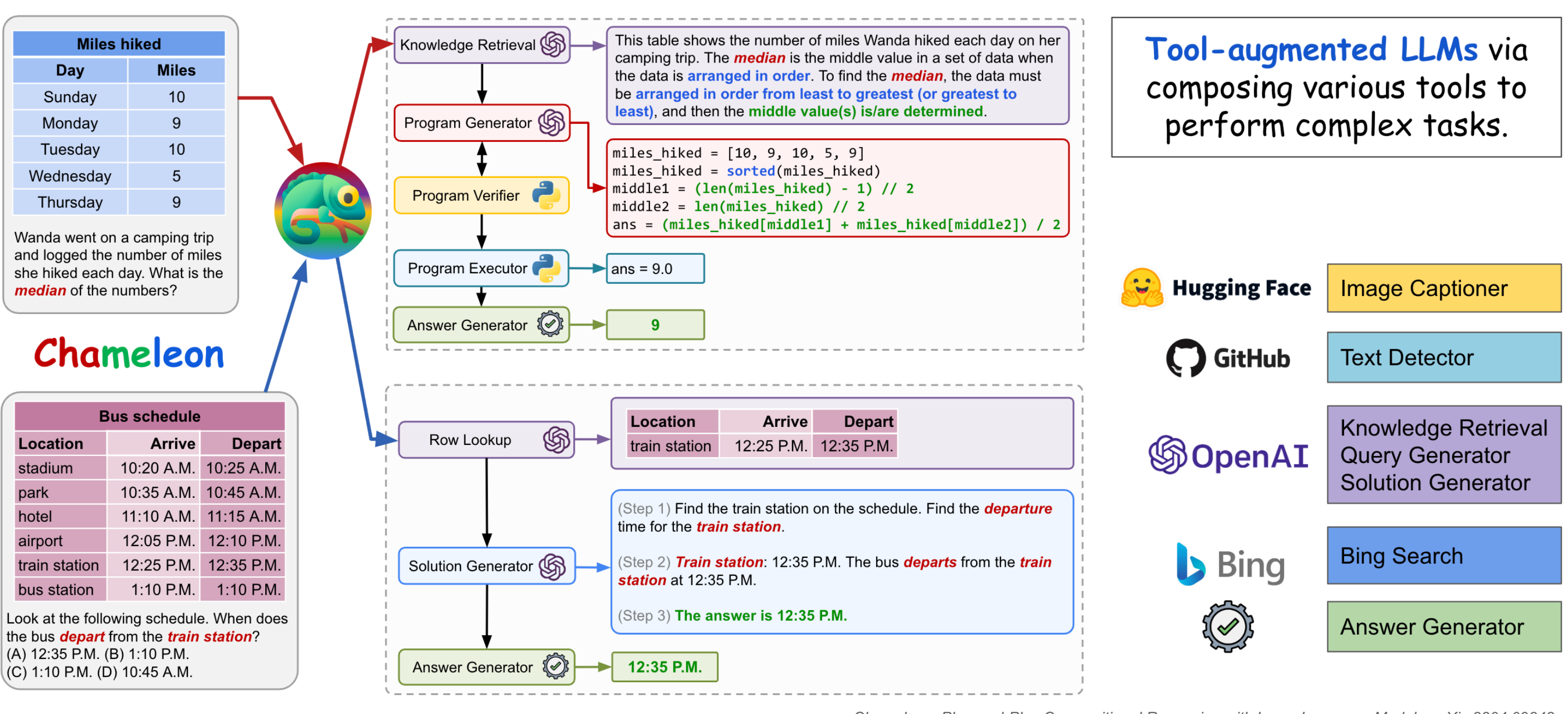**CoT reasoning processes**

(A) Math word problems

An ability is *emergent* if it is not present in smaller models but is present in larger models.

Chain-of-Thought Prompting Elicits Reasoning in Large Language Models, NeurIPS 2022
Emergent Abilities of Large Language Models, Transactions on Machine Learning Research, 2022

Tool-augmented Large Language Models

Chameleon

Tool-augmented LLMs via composing various tools to perform complex tasks.

- Hugging Face — Image Captioner
- GitHub — Text Detector
- OpenAI — Knowledge Retrieval / Query Generator / Solution Generator
- Bing — Bing Search
- Answer Generator

Chameleon: Plug-and-Play Compositional Reasoning with Large Language Models, arXiv:2304.09842

Generalization and Robustness

	T5 (Large)	UnifiedQA (Large)	GPT-3 (text-davinci-002)	GPT-3 (text-davinci-003)
3 balls + 5 balls =	None	5 balls	8 balls	8 balls
23 balls + 145 balls =	None	None	58 balls	168 balls
23 balls + 1,855 balls =	None	None	2,878 balls	2,988 balls

Language models struggle with **large numbers**.

John had 8 balls and he gave 3 to Mary. How many balls does John have now? → Mary has 5 balls. ✓

John had 3 apples. John had 8 balls and he gave 3 to Mary. How many balls does Mary have now? → **Mary has 5 balls.** ✗

John had 8 balls and he gave 4 to Mary. **Who has more balls now?** → **John has more balls.** ✗

GPT-3 (text-davinci-002)

Large language models are **inconsistent** for mathematical reasoning.